Revisitation of the original hot QCD collinear singularity problem

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The long standing issue known as the hot QCD collinear singularity problem has been proven to rely on an incorrect sequence of two mathematical operations. Here, the original derivation of this problem is entirely revisited within the correct sequence, bringing to light new and unexpected conclusions.

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I. INTRODUCTION

The intrinsic non perturbative nature of non zero temperature Quantum field Theories has been recognized for long [1] . Naive thermal perturbation theory can nevertheless be devised, both in imaginary and real time formalisms [2], but then, it promptly appears that, under certain circumstances, the original perturbative series must be re-organized. Such an example of re-organization is provided by the so called *Resummation Program* [3]. This program, RP for short, is a resummation scheme of the leading order thermal fluctuations which, in the literature, are known under the spell of *Hard Thermal Loops*. Whenever one is calculating a physical process related to thermal Green's functions whose external/internal legs are *soft*, it is mandatory to trade the naive thermal perturbation theory for the RP. The softness alluded to above, refers to momenta on the order of the soft scale gT, where T, the temperature, stands for the *hard* scale and g for any relevant (bare/renormalized) and small enough coupling constant.

The RP which has been set up in order to remedy an obvious lack of completeness of the naive thermal perturbation theory, has produced interesting, gauge-invariant results. It is true, however, that it has also met difficulties in the infrared regime of the theories [4, 5].

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One of these difficulties is the sixteen years old hot QCD collinear singularity problem: When one calculates the soft photon emission rate out of a Quark-Gluon Plasma (QGP) at thermal equilibrium, the Resummation Program is in order, but it delivers an answer which is plagued with a collinear singularity [6].

The hot QCD collinear singularity problem has long been quoted an important issue of the Quantum Field Theory non-zero temperature context, and only two years after its discovery, had already become textbook material [6]. It is recorded as a serious obstruction to the high temperature effective Perturbation Theory, and has motivated several attempts of solution [5, 7, 8].

Among the solutions that have been proposed, the latter, in Ref.8, has been adopted widely: It relies on a gauge-invariant introduction of so-called *thermal asymptotic masses* in either bosonic and fermionic sectors of the theories. The thermal asymptotic masses are on the order of the soft scale, gT, and as any mass, they are expected to *screen* the logarithmic collinear singularity under consideration.

However, despite the fact that the introduction of such masses suffers from a lack of justification, the singularity screening it provides reveals itself not efficient enough beyond the stage of a one loop calculation, and indeed, the problem bounces back.

This fate is due to the mechanism of *collinear enhancement*, able to render higher number of loop contributions as important, if no more important than lower number of loop calculations, [9]. Needless to say that in such a dramatic situation, the Resummation Program comes out deprived of any reliability and predictive power, and that, by the time of F. Gelis's thesis striking results, [9], people involved in the matter were almost driven to despair.

This has lead some authors to explore the possibility that extra topologies of graphs be considered that could compensate for the logarithmic collinear singularity of the original diagrams [10]; and one of us, to get back to the original derivation of the hot QCD collinear singularity problem [11].

In this latter instance, [11], it was discovered that the collinear singularity original derivation hinged upon an incorrect sequence of two mathematical steps to be taken, namely, an angular integration followed by a prescription of discontinuity, to proceed along the correct sequence.

In particular, it could be proven that the diagrams involving 1- *effective soft photon-quark-anti-quark vertex*, the other one bare, came out a regular quantity, in contradistinction to

the original derivation where the incorrect sequence was followed: Of course, this could be taken as a serious invitation to revisit the whole problem within the correct sequence.

Unfortunately, the much more involved 2-effective vertex diagram remained an issue because of the incredibly long and difficult entwined angular integrations it entails. However, that issue was the more decisive as the original collinear singularity was explicitly due to that very diagram and to no other. In other words, so long as the 2-effective vertex diagram was not thoroughly calculated within the correct sequence, the hot QCD collinear singularity problem could not be considered a fixed one.

Fixing definitely that issue is the task which is achieved in the present article. The article is organized as follows. Section 2 is a short reminder of the collinear singularity problem met in hot QCD. This section will also serve the purpose of introducing the quantities of interest as well as our notations. In Section 3, the kinematics and the general structure of the calculations involving 1- and 2- effective vertex diagrams are set up.

In order to reach sound conclusions, meticulous calculations of 1- and 2-effective vertex diagrams must be carried out. Such is the case of the former in Section 4, and of the latter in Section 5. Eventually, our conclusions are drawn in Section 6, whereas two appendices complete the article.

Throughout the article, we will be using the convention of upper case letters for quadri-momenta and lower case ones for their components, writing, for example $P=(p_0,\vec{p})$. Our conventions for labelling internal and external momenta can be read off Figure 1.

II. THE COLLINEAR SINGULARITY PROBLEM OF HOT QCD

This sixteen years old issue is the following. The soft real photon emission rate out of a Quark-Gluon Plasma in thermal equilibrium involves the calculation of the quantity

$$\begin{aligned}\Pi_R(Q) = i\int \frac{\mathrm{d}^4P}{(2\pi)^4}(1-2n_F(p_0))\ \mathrm{disc}_{\mathrm{P}}\ Tr\Big\{ &{}^\star S_R(P)\ {}^\star\Gamma_\mu(P_R,Q_R,-P'_A)\\ &{}^\star S_R(P')\ {}^\star\Gamma^\mu(P_R,Q_R,-P'_A)\Big\}\end{aligned} \tag{2.1}$$

The discontinuity is to be taken in the energy variable p_0, by forming the difference of R and A-indiced P-dependent quantities, and within standard notations, fermionic HTL self energies, effective propagators and vertices are respectively given by

$$ {}^\star S_\alpha(P) = \frac{i}{\not P - \Sigma_\alpha(P) + i\epsilon_\alpha p_0} \ , \qquad \alpha = R, A \ , \qquad \epsilon_R = -\epsilon_A = \epsilon \tag{2.2} $$

$$ \Sigma_\alpha(P) = m^2 \int \frac{\mathrm{d}\widehat{K}}{4\pi} \frac{\widehat{\not K}}{\widehat{K}\cdot P + i\epsilon_\alpha} \ , \qquad m^2 = C_F \frac{g^2T^2}{8} \tag{2.3} $$

$$ {}^\star\Gamma_\mu(P_\alpha, Q_\beta, P'_\delta) = -ie\left(\gamma_\mu + \Gamma^{HTL}_\mu(P_\alpha, Q_\beta, P'_\delta)\right) \tag{2.4} $$

$$ \Gamma^{HTL}_\mu(P_\alpha, Q_\beta, P'_\delta) = m^2 \int \frac{\mathrm{d}\widehat{K}}{4\pi} \frac{\widehat{k}_\mu \, \widehat{\not K}}{(\widehat{K}\cdot P + i\epsilon_\alpha)(\widehat{K}\cdot P' + i\epsilon_\delta)} \tag{2.5} $$

where $\widehat{K}$ is the lightlike four vector $(1, \widehat{k})$. As (2.4) is plugged into (2.1), four terms come about, three of them proportional to a collinear singularity. These singular terms are the two terms with one bare vertex γ_μ, the other Γ^{HTL}_μ, plus the term including two HTL vertices, Γ^{HTL}_μ. Thanks to an abelian Ward identity peculiar to the high temperature limit, a partial cancellation of these collinear singularities occurs, but out of the term including two Γ^{HTL}_μ vertices, a collinear singularity remains,

$$ \begin{aligned} -2i\frac{e^2m^2}{q^2}\left(\int \frac{\mathrm{d}\widehat{K}}{4\pi} \frac{1}{\widehat{Q}\cdot\widehat{K} + i\epsilon}\right) \int \frac{\mathrm{d}^4P}{(2\pi)^3} \ \delta(P\cdot\widehat{Q}) \ (1 - 2n_F(p_0)) \\ \times \left[Tr\left({}^\star S_A(P)\widehat{\not Q}\right) - Tr\left({}^\star S_R(P')\widehat{\not Q}\right)\right] \end{aligned} \tag{2.6} $$

where, the soft photon being real, Q is the lightlike 4-vector $Q = q\widehat{Q} = q(1, \widehat{q})$, with q a real positive number. In the literature, this result is ordinarily written in the form

$$ \frac{C^{st}}{\varepsilon} \int \frac{d^4P}{(2\pi)^4} \ \delta(\widehat{Q}\cdot P) \ (1 - 2n_F(p_0)) \sum_{s=\pm1, V=P,P'} \pi(1 - s\frac{v_0}{v})\beta_s(V) \tag{2.7} $$

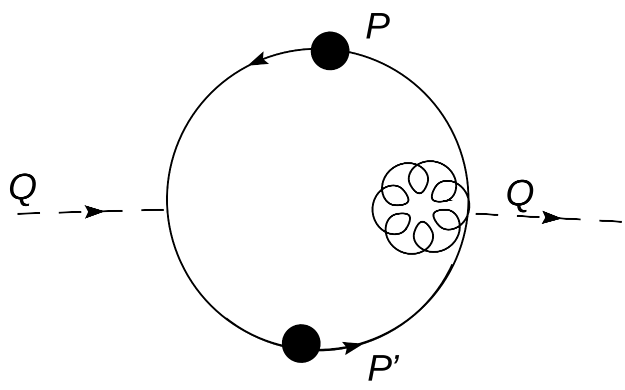
where the overall $1/\varepsilon$ comes from a dimensionally regularized evaluation of the factored out angular integration appearing in (2.6), and where $\beta_s(V)$ is related to the effective fermionic propagator usual parametrization [6],

$$ {}^\star S(P) = \frac{i}{2} \sum_{s=\pm1} \widehat{\not P}_s \, {}^\star\Delta^s(p_0, p) \tag{2.8} $$

where $\widehat{P_s} = (1, s\widehat{p})$, the label s referring to the two dressed fermion propagating modes. One has

$$ {}^\star\Delta^s(p_0, p) = \left(p_0 - sp - \frac{m^2}{2p}[(1 - s\frac{p_0}{p}) \ln \frac{p_0 + p}{p_0 - p} + 2s]\right)^{-1} \tag{2.9} $$

FIG. 1: Self energy diagram involving two effective propagators and one vertex HTL correction



the two Retarded/Advanced solutions corresponding to

$$^\star\Delta^s_\alpha(p_0,p) \equiv {}^\star\Delta^s(p_0+i\epsilon_\alpha,p) = \alpha_s(p_0,p) - i\pi\epsilon(\epsilon_\alpha)\beta_s(p_0,p), \quad \epsilon_R = -\epsilon_A = 1 \tag{2.10}$$

where $\epsilon(x)$ is the distribution "sign of x", and $\alpha = R, A$.

III. 1- AND 2-EFFECTIVE VERTEX CONTRIBUTIONS AND KINEMATICS

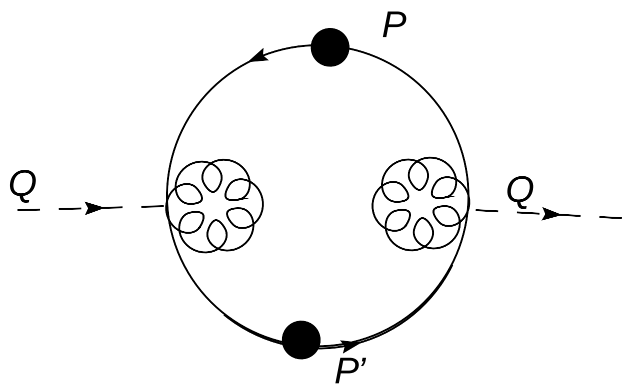
The historical derivation just reminded above, however, is plagued with erroneous manipulations that have been put forth in Ref.11. In the R/A formalism, the two diagrams including one bare vertex γ_μ, the other Γ^{HTL}_μ, (2.5), lead to the expression

$$\begin{aligned}\Pi_R^{(\star,\star;1)}(Q) = -ie^2m^2 \int \frac{\mathrm{d}^4P}{(2\pi)^4}(1-2n_F(p_0))\\ \mathrm{disc}_{p_0}\int \frac{\mathrm{d}\widehat{K}}{4\pi}\frac{Tr\left({}^\star S_R(P)\widehat{\not K}{}^\star S_R(P')\widehat{\not K}\right)}{(\widehat{K}\cdot P+i\epsilon)(\widehat{K}\cdot P'+i\epsilon)}\end{aligned} \tag{3.1}$$

where the superscript $(\star,\star;1)$ in the left hand side refers to a self energy diagram involving two effective propagators and one vertex HTL correction, as depicted in Figure 1.

It is this diagram which is now being analyzed within the correct sequence of Eq.(3.1), where the angular average is to be performed before the discontinuity in p_0 is taken. One

FIG. 2: Self energy diagram involving two effective propagators and two vertex HTL corrections



gets

$$\Pi_R^{(\star,\star;1)}(Q) = 2ie^2m^2 \int \frac{\mathrm{d}^4P}{(2\pi)^4}(1-2n_F(p_0))$$
$$\sum_{s,s'=\pm1} {}^\star\Delta_R^{s'}(P')\Big\{-2i\pi\beta_s(P)\int \frac{\mathrm{d}\widehat{K}}{4\pi}\frac{\widehat{K}\cdot\widehat{P}_s}{\widehat{K}\cdot\widehat{P}+i\epsilon}\frac{\widehat{K}\cdot\widehat{P}'_{s'}}{\widehat{K}\cdot\widehat{P}'+i\epsilon}$$
$$+{}^\star\Delta_R^{s}(P)\mathrm{disc}_{\mathrm{p_0}}\int \frac{\mathrm{d}\widehat{K}}{4\pi}\frac{\widehat{K}\cdot\widehat{P}_s}{\widehat{K}\cdot\widehat{P}+i\epsilon}\frac{\widehat{K}\cdot\widehat{P}'_{s'}}{\widehat{K}\cdot\widehat{P}'+i\epsilon}\Big\} \tag{3.2}$$

where we have used $\mathrm{disc}_{p_0}{}^\star\Delta_R^s(P) = -2i\pi\beta_s(P)$, whereas a factor of 2 accounts for the two 1-effective vertex diagrams, which contribute equally. Defining $W^{(1)}(P,P')$ the function

$$W^{(1)}(P,P') = \int \frac{\mathrm{d}\widehat{K}}{4\pi}\frac{\widehat{K}\cdot\widehat{P}_s}{\widehat{K}\cdot\widehat{P}+i\epsilon}\frac{\widehat{K}\cdot\widehat{P}'_{s'}}{\widehat{K}\cdot\widehat{P}'+i\epsilon} \tag{3.3}$$

one obtains for the imaginary part, the expression

$$\mathrm{Im}\ \Pi_R^{(\star,\star;1)}(Q) = 2\pi e^2m^2 \int \frac{\mathrm{d}^4P}{(2\pi)^4}(1-2n_F(p_0))\sum_{s,s'=\pm1}\Big\{-2\pi\beta_s(P)\beta_{s'}(P')W^{(1)}(P,P')$$
$$+\left(\alpha_s(P)\beta_{s'}(P')+\alpha_{s'}(P')\beta_s(P)\right)\left(-i\ \mathrm{disc}_{p_0}W^{(1)}(P,P')\right)\Big\} \tag{3.4}$$

For the diagram of Figure 2, involving two effective vertex,

$$\Pi_R^{(\star,\star;2)}(Q) = -ie^2m^4 \int \frac{\mathrm{d}^4P}{(2\pi)^4}(1-2n_F(p_0))$$
$$\mathrm{disc}_{p_0}\int \frac{\mathrm{d}\widehat{K}}{4\pi}\int \frac{\mathrm{d}\widehat{K}'}{4\pi}\widehat{K}\cdot\widehat{K}'\frac{Tr\left({}^\star S_R(P)\widehat{K}\!\!\!/\,{}^\star S_R(P')\widehat{K}\!\!\!/'\right)}{(\widehat{K}\cdot P+i\epsilon)(\widehat{K}\cdot P'+i\epsilon)(\widehat{K}'\cdot P+i\epsilon)(\widehat{K}'\cdot P'+i\epsilon)} \tag{3.5}$$

so that, defining $W^{(2)}(P,P')$ the function

$$W^{(2)}(P,P') = \int \frac{\mathrm{d}\widehat{K}}{4\pi} \int \frac{\mathrm{d}\widehat{K}'}{4\pi}\ \widehat{K}\cdot\widehat{K}' \frac{\widehat{K}\cdot\widehat{P}_s\ \widehat{K}'\cdot\widehat{P}'_{s'} + \widehat{K}\cdot\widehat{P}'_{s'}\ \widehat{K}'\cdot\widehat{P}_s - \widehat{K}\cdot\widehat{K}'\widehat{P}_s\cdot\widehat{P}'_{s'}}{(\widehat{K}\cdot P + i\epsilon)(\widehat{K}\cdot P' + i\epsilon)(\widehat{K}'\cdot P + i\epsilon)(\widehat{K}'\cdot P' + i\epsilon)} \tag{3.6}$$

one gets for the imaginary part, an analogous expression of

$$\begin{aligned}\mathrm{Im}\ \Pi_R^{(\star,\star;2)}(Q) = \pi e^2 m^4 \int \frac{\mathrm{d}^4 P}{(2\pi)^4}(1-2n_F(p_0)) \sum_{s,s'=\pm1} \Big\{ -2\pi\beta_s(P)\beta_{s'}(P')W^{(2)}(P,P') \\ + \left(\alpha_s(P)\beta_{s'}(P') + \alpha_{s'}(P')\beta_s(P)\right)\left(-i\ \mathrm{disc}_{p_0} W^{(2)}(P,P')\right)\Big\}\end{aligned} \tag{3.7}$$

whose structure, the same as in the case of a single effective vertex insertion, (3.4), allows some common and generic treatment of either cases : In both (3.4) and (3.7), one has to cope with $\beta_s(P)$-distributions standing for the sum of a *pole* part and a *cut* part. Writing $\beta_s \equiv \beta_s^{(p)} + \beta_s^{(c)}$, the textbook expressions are, [6],

$$-\beta_s^{(p)}(p_0,p) = Z_s(p)\delta\left(p_0 - \omega_s(p)\right) + Z_{-s}(p)\delta\left(p_0 + \omega_{-s}(p)\right) \tag{3.8}$$

$$-\beta_s^{(c)}(p_0,p) = \frac{m^2}{2p} \frac{(1-s\frac{p_0}{p})\Theta(-P^2)}{\left(p(1-s\frac{p_0}{p}) - \frac{m^2}{2p}\left((1-s\frac{p_0}{p})\ln|\frac{p_0+p}{p_0-p}| + 2s\right)\right)^2 + \frac{\pi^2 m^4}{4p^2}(1-s\frac{p_0}{p})^2} \tag{3.9}$$

where for all $s=\pm1$ and all p, the $Z_s(p)$ stand for the residues at the quasi-particle poles. Now, in view of (3.8) and (3.9), three types of contribution to (3.4) and (3.7) have to be considered :

(i) Contributions involving the product of distributions $\beta_s^{(p)}(P)$ and $\beta_{s'}^{(p)}(P')$,

(ii) crossed contributions involving the product of distributions $\beta_s^{(p)}(P)$ and $\beta_{s'}^{(c)}(P')$,

(iii) contributions involving the product of distributions $\beta_s^{(c)}(P)$ and $\beta_{s'}^{(c)}(P')$.

In case (i), there is no infrared singularity problem at all, because none of the quantities P^2, P'^2, $2Q\cdot P$ and $p_0^2 - 2pxp_0 + p^2$ can ever vanish over the whole integration range. This will be exemplified to a large extent in the sequel. Case (iii) has been studied thoroughly and shown to lead to singularity free contributions, [12].

The intermediate, crossed case (ii) remains to be studied. The two crossed possibilities contribute equally and the crossed term $\beta_{s'}^{(p)}(p'_0,p') \times \beta_s^{(c)}(p_0,p)$ comes out to be proportional to the product of distributions

$$\Theta(-P^2) \times \left\{Z_{s'}(p')\delta\left(p'_0 - \omega_{s'}(p')\right) + Z_{-s'}(p)\delta\left(p'_0 + \omega_{-s'}(p')\right)\right\} \tag{3.10}$$

where the residues at the quasi-particle poles read as

$$Z_s(p) = \frac{\omega_s^2(p) - p^2}{2m^2} \tag{3.11}$$

The second delta is clearly incompatible with the overall $\Theta(-P^2)$: It would require that P'^2 be strictly positive, whereas it fixes a strictly negative term of $2Q{\cdot}P$. Since $P'^2 = P^2 + 2Q{\cdot}P$, this is impossible to satisfy at $P^2 \leq 0$. There is no incompatibility with the first delta function which fixes P'^2 and $2Q \cdot P$ at strictly positive values, whereas P^2 can reach zero from below.

The constraint of $\delta(p_0' - \omega_{s'}(p'(x)))$ is common to both contributions appearing inside the curly brackets of either (3.4) or (3.7). Defining $p'(x) = \sqrt{q^2 + 2pqx + p^2}$, with the two cosines $x = \widehat{qp}$, and $y = \widehat{qp'}$, one has $2Q{\cdot}P = 2Q{\cdot}P' = 2q(\omega_{s'}(p'(x)) - yp')$. Since $-1 \leq y \leq 1$, and since $\omega_{s'}^2(p') - p'^2 > 0$,

one can deduce that

$$-1 \leq x < \frac{p_0}{p} \tag{3.12}$$

Then, since $P'^2 = \omega_{s'}^2(p'(x)) - p'^2(x) > 0$, so is therefore $P^2 + 2Q \cdot P$, which gives

$$\frac{p_0^2 - p^2 + 2qp_0}{2qp} > x \tag{3.13}$$

and so

$$1 + \frac{p_0^2 - p^2 + 2qp_0}{2qp} = \frac{(p_0 + p)(2q + p_0 - p)}{2qp} > 1 + x \geq 0 \tag{3.14}$$

The kinematics inherited from this common constraint restrict the integration domain to the boundaries

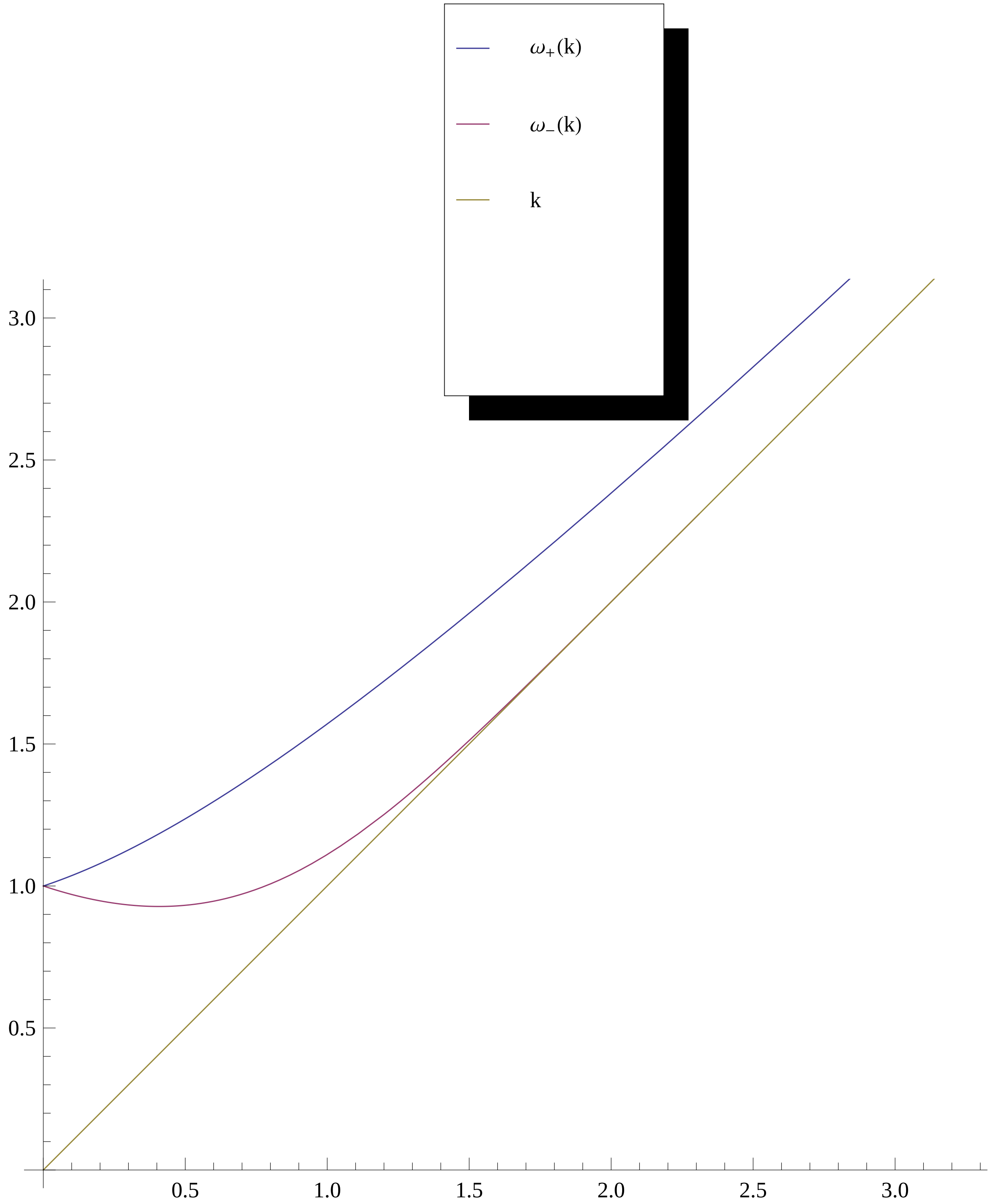
$$\mathcal{O}(q) = \mathcal{O}(p) = m \ , \quad p \leq q \ , \quad -p < p_0 \ , \quad -1 \leq x < \frac{p_0}{p} \tag{3.15}$$

For the first terms in the curly brackets of both (3.4) and (3.7), proportional to the products $\beta_{s'}^{(p)}(p_0', p') \times \beta_s^{(c)}(p_0, p)$, an extra constraint of $\Theta(-P^2)$ comes into play in view of (3.9), and modifies (3.15) into an integration domain bounded by the relations

$$\mathcal{O}(q) = \mathcal{O}(p) = m \ , \quad p \leq q \ , \quad -p < p_0 \leq p \ , \quad -1 \leq x < \frac{p_0}{p} \tag{3.16}$$

Actually, the arguments developed after (3.9) do not apply to the second terms in the curly brackets of (3.4) and (3.7), the ones proportional to $\alpha_s(P) \times \beta_{s'}^{(p)}(P')$-contributions; but it turns out that the terms of $\mathrm{disc}_{p_0} W^{(i)}(P, P')$, for $i \in \{1, 2\}$, effectively restore the previous

FIG. 3: Dispersion relations : $\omega_{\pm}/\mathrm{m}$ as functions of k/m



$\Theta(-P^2)$-constraint, as can be read off (5.1.23) and (5.2.37) below, so as to make of (3.16) the effective integration domain of the required resummation.

Note that these inequalities automatically preclude any risk of collinear singularity at $x = +1$, but not at $x = -1$, where the collinear singularity was historically located [5].

IV. 1- EFFECTIVE VERTEX CALCULATIONS

This case is given by Eq.(3.4), with the angular function $W^{(1)}$ given by (3.3). In this case, the explicit calculation is quite simple. One gets

$$W^{(1)}(P,P') = \frac{ss'}{pp'} + \frac{s'}{p'}(1-s\frac{p_0}{p})\frac{1}{2p}\ln\frac{p_0+p}{p_0-p} + \frac{s}{p}(1-s'\frac{p'_0}{p'})\frac{1}{2p'}\ln\frac{p'_0+p'}{p'_0-p'} \\ +(1-s\frac{p_0}{p})(1-s'\frac{p'_0}{p'})\frac{1}{2Q\cdot P}\ln\frac{P'^2}{P^2} \quad (4.1)$$

and so

$$\mathrm{disc}_{\mathrm{p_0}} W^{(1)}(P,P') = -i\pi\Theta(-P^2)(1-s\frac{p_0}{p})\left\{\frac{s'}{pp'} + (1-s'\frac{p'_0}{p'})\frac{\varepsilon(p_0)}{Q\cdot P}\right\} \quad (4.2)$$

The imaginary part of $\Pi_R^{(\star,\star;1)}(Q)$ can accordingly be written as the full expression

$$\mathrm{Im}\ \Pi_R^{(\star,\star;1)}(Q) = -4\pi^2 e^2 m^2 \int \frac{\mathrm{d}^4P}{(2\pi)^4}(1-2n_F(p_0)) \sum_{s,s'=\pm1} \beta_s^{(c)}(P)\beta_{s'}^{(p)}(P') \\ \left\{\frac{ss'}{pp'}+\frac{s'}{p'}(1-s\frac{p_0}{p})\frac{1}{2p}\ln\frac{p_0+p}{p_0-p}+\frac{s}{p}(1-s'\frac{p'_0}{p'})\frac{1}{2p'}\ln\frac{p'_0+p'}{p'_0-p'}+(1-s\frac{p_0}{p})(1-s'\frac{p'_0}{p'})\frac{1}{2Q\cdot P}\ln\frac{P'^2}{P^2}\right\} \\ -4\pi^2 e^2 m^2 \int \frac{\mathrm{d}^4P}{(2\pi)^4}(1-2n_F(p_0))\Theta(-P^2) \\ \sum_{s,s'=\pm1} \alpha_s(P)\beta_{s'}^{(p)}(P')(1-s\frac{p_0}{p})\left\{\frac{s'}{pp'} + (1-s'\frac{p'_0}{p'})\frac{\varepsilon(p_0)}{Q\cdot P}\right\} \quad (4.3)$$

and is to be integrated over the domain (3.16).

Let us begin with focusing on the first curly bracket of (4.3) : Because P'^2 as well as $2Q\cdot P$ are strictly positive, only the logarithm of $p_0 - p$ in the integrand, is able to yield a diverging behavior, and there are two of them. Such a potentially dangerous behavior is for example the one of

$$+2\pi e^2 \sum_{s,s'=\pm1} \int \frac{\mathrm{d}^3p}{(2\pi)^3} \int_{-p}^{+p} \frac{\mathrm{d}p_0}{2\pi}(1-2n_F(p_0))\ \frac{1}{2Q\cdot P}\ln\frac{(p_0-p)(p_0+p)}{P'^2} \\ \times\ \beta_s^{(c)}(p_0,p)(\omega_{s'}^2(p') - p'^2)\delta\left(p'_0 - \omega_{s'}(p')\right)(1-s\frac{p_0}{p})(1-s'\frac{p'_0}{p'}) \quad (4.4)$$

where (3.11) has been used. However, at $s = +1$, the logarithmic divergence of the integrand is suppressed by a factor of $1 - p_0/p$ in $\beta_+^{(c)}(p_0, p)$, so that the case of $s = -1$ only must be considered whose potentially singular part reads

$$+2\pi e^2 \int \frac{p^2 \mathrm{d}p}{(2\pi)^2} \int_{-p}^{+p} \frac{\mathrm{d}p_0}{2\pi} (1 - 2n_F(p_0)) \; (1 + \frac{p_0}{p}) \; \beta_-^{(c)}(p_0, p) \; \ln \frac{p - p_0}{p}$$
$$\times \sum_{s'=\pm 1} \int_{-1}^{\frac{p_0}{p}} \mathrm{d}x \; \frac{\delta\left(q + p_0 - \omega_{s'}(p'(x))\right)}{2q(\omega_{s'}(p'(x)) - (q + px))} \; (\omega_{s'}^2(p'(x)) - p'^2(x))(1 - s' \frac{\omega_{s'}(p'(x))}{p'(x)}) \tag{4.5}$$

Because $\omega_{s'}(p'(x))$ is a fairly complicated, implicit function of x, the last line of (4.5) is certainly hard to get exactly. Fortunately this is not necessary either : It is sufficient that, in a neighborhood of $p_0 = p$, the second line of (4.5) defines a regular function of p_0, say $F(p_0)$. This condition is met indeed, and since this situation is generic of all the potentially singular behaviors attached to logarithms of $(p - p_0)$, a proof is sketched in Appendix A. Then, in order to isolate the potentially singular behavior of (4.5), one may re-write the second line of (4.5) as the sum $[F(p_0) - F(p)] + F(p)$. Whereas the first term, $[F(p_0) - F(p)]$ annihilates the potentially divergent behavior of the logarithms, $[\ln(p - p_0)/p]^c$, the second, $F(p)$, gives a contribution proportional to the would be singular part of (4.5), that is to

$$\int^p \mathrm{d}p_0 \; \frac{\ln \frac{p-p_0}{p}}{\ln^2 \frac{p-p_0}{p}} \sim \lim_{p_0 = p} Li(\frac{p - p_0}{p}) = \lim_{p_0 = p} \frac{p - p_0}{p} \int_1^\infty \frac{1}{x^2} \, \frac{\mathrm{d}x}{\ln x + \ln \frac{p}{p - p_0}} = 0 \tag{4.6}$$

where (3.9) has been used, and where $Li(x)$ is the *Logarithm-integral function of* x, [13].

For the second term of (4.3), the one involving the discontinuity in p_0, it is immediate to see that the same arguments apply, over the same integration range (3.16), with the same conclusion.

Eventually, in contradistinction with the *historical* improper derivation, the imaginary part of $\Pi_R^{(\star,\star;1)}(Q)$ comes out singularity free when evaluated along the correct sequence of discontinuity and angular average operations.

V. 2-EFFECTIVE VERTEX CALCULATIONS

Though crucial, since the original collinear singularity is explicitly due to it, this case is far more difficult because, as an unavoidable step, the angular function $W^{(2)}(P, P')$ of (3.6) must be known exactly. Let us begin with recalling this function

$$W^{(2)}(P, P') = \int \frac{\mathrm{d}\widehat{K}}{4\pi} \int \frac{\mathrm{d}\widehat{K}'}{4\pi} \; \widehat{K} \cdot \widehat{K}' \frac{\widehat{K} \cdot \widehat{P}_s \; \widehat{K}' \cdot \widehat{P}'_{s'} + \widehat{K} \cdot \widehat{P}'_{s'} \; \widehat{K}' \cdot \widehat{P}_s - \widehat{K} \cdot \widehat{K}' \widehat{P}_s \cdot \widehat{P}'_{s'}}{(\widehat{K} \cdot P + i\epsilon)(\widehat{K} \cdot P' + i\epsilon)(\widehat{K}' \cdot P + i\epsilon)(\widehat{K}' \cdot P' + i\epsilon)} \tag{3.6}$$

and define $W^{(2)}(P,P') = W_1^{(2)}(P,P') + W_2^{(2)}(P,P')$. The function $W_1^{(2)}(P,P')$ corresponds to the first two terms in the numerator of (3.6). They are symmetric in the exchange of P and P' and contribute equally. That is,

$$W_1^{(2)}(P,P') = 2\int_{\widehat{K}}\int_{\widehat{K}'} \frac{\widehat{K}\cdot\widehat{K}'}{(\widehat{K}\cdot P' + i\epsilon)(\widehat{K}'\cdot P + i\epsilon)} \left\{ \frac{ss'}{pp'} + \frac{s'}{p'}(1 - s\frac{p_0}{p})\frac{1}{\widehat{K}\cdot P + i\epsilon} \right.$$
$$\left. + \frac{s}{p}(1 - s'\frac{p'_0}{p'})\frac{1}{\widehat{K}'\cdot P' + i\epsilon} + (1 - s\frac{p_0}{p})(1 - s'\frac{p'_0}{p'})\frac{1}{(\widehat{K}\cdot P + i\epsilon)(\widehat{K}'\cdot P' + i\epsilon)} \right\} \qquad (5.1)$$

whereas $W_2^{(2)}(P,P')$ is the function

$$W_2^{(2)}(P,P') = -\widehat{P}_s\cdot\widehat{P}'_{s'}\int_{\widehat{K}}\int_{\widehat{K}'} \frac{(\widehat{K}\cdot\widehat{K}')^2}{(\widehat{K}\cdot P + i\epsilon)(\widehat{K}\cdot P' + i\epsilon)(\widehat{K}'\cdot P + i\epsilon)(\widehat{K}'\cdot P' + i\epsilon)} \qquad (5.2)$$

We now cope exclusively with $W_1^{(2)}(P,P')$. The calculation of $W_2^{(2)}(P,P')$ being "orders of magnitude" more difficult will be dealt with in subsection 5.2.

A. The case of $W_1^{(2)}(P,P')$

The contribution of $W_1^{(2)}(P,P')$ to Im $\Pi_R^{(\star,\star;2)}(Q)$, is obtained by substituting $W_1^{(2)}(P,P')$ for $W^{(2)}(P,P')$ in (3.7), and we begin with the $\beta_{s'}^{(p)}(P') \times \beta_s^{(c)}(P)$ -term.

- From (5.1), a first part, coming from the term ss'/pp' contributes to (3.7) the amount

$$-2\pi^2 e^2 \int_P (1 - 2n_F(p_0)) \sum_{s,s'=\pm1} \beta_s^{(c)}(p_0,p)\ \frac{\omega_{s'}^2(p') - p'^2}{2m^2}\delta\left(p'_0 - \omega_{s'}(p')\right)$$
$$\times\frac{ss'}{pp'}\ \Sigma_R(P)\cdot\Sigma_R(P') \qquad (5.1.3)$$

where the "self energy four-vector" has components,

$$\Sigma_\alpha^0(P) = \frac{m^2}{p}Q_0(\frac{p_0}{p})\ ,\qquad \Sigma_\alpha^i(P) = (\frac{\vec{p}^i}{p} \equiv \widehat{p}^i)\frac{m^2}{p}Q_1(\frac{p_0}{p}) \qquad (5.1.4)$$

with Q_0 and Q_1, the Legendre functions

$$Q_1(x) = xQ_0(x) - 1\ ,\qquad Q_0(x) = \frac{1}{2}\ln\frac{x+1}{x-1} \qquad (5.1.5)$$

The label $\alpha = \{R, A\}$ keeps on denoting one of the two *Retarded* or *Advanced* specifications of the real time formalism being used, and in the right hand sides of (5.1.4) these specifications are encoded in the logarithmic determinations. Because of the delta distribution, $\delta\left(p'_0 - \omega_{s'}(p')\right)$, one of the self energies of (5.1.3) is the regular function, $\Sigma_R(\omega_{s'}(p'), p')$,

whereas the other one, $\Sigma_R(p_0,p)$, entails the logarithmic components of (5.1.4) and (5.1.5). Over the integration range of $-p < p_0 \leq +p$, though themselves divergent, but logarithmically only, these components lead to the same singularity free result as obtained in Section 4, Eq.(4.6).

- For the second term in the big parenthesis of (5.1) one can take advantage of Eq.(4.14) of Ref.12, to find

$$\frac{s'}{p'}(1-s\frac{p_0}{p})\int_{\widehat{K}}\int_{\widehat{K}'}\frac{\widehat{K}\cdot\widehat{K}'}{(\widehat{K}\cdot P'+i\epsilon)(\widehat{K}'\cdot P+i\epsilon)}\,\frac{1}{\widehat{K}\cdot P+i\epsilon}=\frac{s'}{p'}(1-s\frac{p_0}{p})\Big\{\frac{1}{p^2}Q_1(\frac{p_0}{p})\frac{1}{2p'}\ln\frac{p'_0+p'}{p'_0-p'}$$
$$+\frac{1}{p}\left(\frac{p_0}{p}-\frac{P^2}{p^2}Q_0(\frac{p_0}{p})\right)\frac{1}{2Q\cdot P}\ln\frac{P'^2}{P^2}\Big\} \qquad (5.1.6)$$

so that, when plugged back into (3.7), one gets

$$-2\pi^2e^2m^4\int\frac{\mathrm{d}^3p}{(2\pi)^3}\int_{-p}^{+p}\frac{\mathrm{d}p_0}{2\pi}(1-2n_F(p_0))\sum_{s,s'=\pm1}\beta_s^{(c)}(p_0,p)\,\frac{\omega_{s'}^2(p')-p'^2}{2m^2}\delta\left(p'_0-\omega_{s'}(p')\right)$$
$$\times\frac{s'}{p'}(1-s\frac{p_0}{p})\Big\{\frac{1}{p^2}Q_1(\frac{p_0}{p})\frac{1}{2p'}\ln\frac{\omega_{s'}(p')+p'}{\omega_{s'}(p')-p'}$$
$$+\frac{1}{p}\left(\frac{p_0}{p}-\frac{P^2}{p^2}Q_0(\frac{p_0}{p})\right)\frac{1}{2q(\omega_{s'}(p')-p'y)}\ln\frac{\omega_{s'}^2(p')-p'^2}{P^2}\Big\} \qquad (5.1.7)$$

where we have used $2Q\cdot P = 2Q\cdot P' = 2q(\omega_{s'}(p')-p'y) > 0$, in order to emphasize the non vanishing character of this factor. Again, the integrand "wildest behavior" is the one of the logarithms of $p_0 - p$, which, integrated over the interval $-p < p_0 \leq +p$ leads to regular contributions.

- The same conclusion holds for the third term in the big parenthesis of (5.1), which, easily obtained out of the second one, is quoted here for the sake of completeness,

$$-2\pi^2e^2m^4\int\frac{\mathrm{d}^3p}{(2\pi)^3}\int_{-p}^{+p}\frac{\mathrm{d}p_0}{2\pi}(1-2n_F(p_0))\sum_{s,s'=\pm1}\beta_s^{(c)}(p_0,p)\,\frac{\omega_{s'}^2(p')-p'^2}{2m^2}\delta\left(p'_0-\omega_{s'}(p')\right)$$
$$\times\frac{s}{p}(1-s'\frac{p'_0}{p'})\Big\{\frac{1}{p'^2}Q_1(\frac{\omega_{s'}(p')}{p'})\frac{1}{2p}\ln\frac{p_0+p}{p_0-p}$$
$$+\frac{1}{p'}\left(\frac{\omega_{s'}(p')}{p}-\frac{P'^2}{p'^2}Q_0(\frac{\omega_{s'}(p')}{p'})\right)\frac{1}{2q(\omega_{s'}(p')-p'y)}\ln\frac{\omega_{s'}^2(p')-p'^2}{P^2}\Big\} \qquad (5.1.8)$$

Note that the last term of (5.1.8) is not induced by an error of *copy and paste*, but reflects the symmetry of $1/2Q{\cdot}P\times\ln P'^2/P^2$ under the exchange of P' and P, since $2Q{\cdot}P = P'^2 - P^2$.

- With the fourth term in the big parenthesis of (5.1), things become more involved. This term in effect, entails the following angular integration

$$(1 - s\frac{p_0}{p})(1 - s'\frac{p'_0}{p'}) \int_{\widehat{K}} \int_{\widehat{K}'} \frac{\widehat{K}\cdot\widehat{K}'}{(\widehat{K}\cdot P + i\epsilon)(\widehat{K}\cdot P' + i\epsilon)(\widehat{K}'\cdot P + i\epsilon)(\widehat{K}'\cdot P' + i\epsilon)} \tag{5.1.9}$$

One can take advantage of the calculations of Ref.12, in particular of the angular identity ($R = (r_0, \vec{r})$, $r = |\vec{r}|$)

$$\int \frac{\mathrm{d}\widehat{K}}{4\pi} \frac{\widehat{K}^i}{(\widehat{K}\cdot R + i\epsilon)^2} = \frac{r^i}{r^2}\left(\frac{1}{2r}\ln\frac{r_0 + r}{r_0 - r} - \frac{r_0}{R^2 + i\epsilon r_0}\right) \tag{5.1.10}$$

an euclidean version of which can be found in [6]. Using it, the result can be cast into the form

$$\int \frac{\mathrm{d}\widehat{K}}{4\pi} \frac{\widehat{K}^\mu}{(\widehat{K}\cdot P + i\epsilon)(\widehat{K}\cdot P' + i\epsilon)} \int \frac{\mathrm{d}\widehat{K}'}{4\pi} \frac{\widehat{K}'_\mu}{(\widehat{K}'\cdot P + i\epsilon)(\widehat{K}'\cdot P' + i\epsilon)} = - \sum_{i,j=0}^{3} \left(\sum_{k=-2}^{+1} a_{ij}^k \, (2Q\cdot P)^k\right) F_i F_j \tag{5.1.11}$$

where the following definitions are used :

$$r^2(s) = p^2 + 2pqxs + q^2 s^2 \ , \qquad R^2(s) = P^2 + Zs \ , \qquad Z = 2Q\cdot P \tag{5.1.12}$$

and where the $F'_i s$ stand for the four functions

$$F_0(P,Q) = \int_0^1 \frac{\mathrm{d}s}{R^2(s)} = \frac{1}{2Q\cdot P}\ln\frac{P'^2}{P^2} \tag{5.1.13}$$

$$F_2(P,Q) = \int_0^1 \frac{\mathrm{d}s}{r^2(s)} = \frac{1}{qp\sqrt{1-x^2}} \arctan\frac{q\sqrt{1-x^2}}{p + qx} \tag{5.1.14}$$

$$F_1 - \frac{px}{q}F_2 = \int_0^1 \frac{s\mathrm{d}s}{r^2(s)} = \frac{1}{2q^2}\ln\frac{p'^2}{p^2} - \frac{px}{q}F_2 \tag{5.1.15}$$

$$F_3(P,Q) = \int_0^1 \frac{\mathrm{d}s}{r^2(s)R^2(s)} = \frac{1}{q^2\ (p_0^2 - 2pxp_0 + p^2)^2}\Big\{(q^2P^2 - pqxZ)F_2 + Z^2F_0 - q^2 Z F_1\Big\} \tag{5.1.16}$$

Eventually, the non vanishing a_{ij}^k-coefficients of (5.1.11) are polynomials in p_0

$$a_{22}^{-2} = -q^2P^2, \ a_{22}^{-1} = qp_0 \tag{5.1.17}$$

$$a_{33}^{-2} = -q^2(P^2)^3,\ a_{33}^{-1} = \frac{5}{2}qp_0(P^2)^2,\ a_{33}^{0} = -\frac{9}{4}(P^2)^2 - \frac{5}{2}p^2P^2,\ a_{33}^{1} = \frac{p_0(3P^2+4p^2)}{4q} \quad (5.1.18)$$

$$a_{02}^{0} = 1 \quad (5.1.19)$$

$$a_{03}^{-1} = -qp_0P^2,\ a_{03}^{0} = \frac{3}{2}P^2,\ a_{03}^{1} = -\frac{p_0}{q} \quad (5.1.20)$$

$$a_{23}^{-2} = 2q^2(P^2)^2,\ a_{23}^{-1} = -4qp_0P^2,\ a_{23}^{0} = \frac{11}{4}P^2 + \frac{3}{2}p^2,\ a_{23}^{1} = -\frac{p_0}{q} \quad (5.1.21)$$

Over the integration domain (3.16), since $Z = 2Q{\cdot}P$ does not vanish, the potentially singular behaviors of (3.7) are to be looked for in relation to the behaviors of the $F_i's$.

- The case of $(F_2)^2$-contributions, with associated coefficients (5.1.17), is dealt with easily. Since F_2 is a perfectly regular function of its variables, $(F_2)^2$-contributions to (3.7) are singularity free.

- And so is, in the same vein, the F_0F_2-contribution to (3.7), corresponding to the coefficient (5.1.19).

- For the function F_3, one has a denominator of $(p_0^2 - 2pxp_0 + p^2)^2$ which has no zeros in the integration range (3.16). The potentially singular most behavior of F_3 is again the one of F_0, with its $\ln[(p_0-p)/p]$-term. It results that not only contributions to (3.7) of type (F_2F_3), with associated coefficients (5.1.21), but also $(F_3)^2$- and (F_0F_3)-contributions to (3.7), respectively associated to coefficients (5.1.18) and (5.1.20), are singularity-free.

The $W_1^{(2)}(P,P')$ -contributions to the 2-effective vertex part of the soft photon emission rate involve another piece, the one associated to the term $-i\mathrm{disc}_{p_0}W_1^{(2)}(P,P')$.

As made clear by a simple inspection of the p_0-dependences in (5.1.3), (5.1.7) and (5.1.8), taking the discontinuity in p_0 just amounts to substitute a term of $\pm i\pi\Theta(-P^2)$ for a logarithmic term of $\ln[(p_0-p)/p]$, all of the other discontinuities being zero or giving zero: Such is for example the case of the discontinuity proportional to $\delta(2Q{\cdot}P)$ which has no support in (3.16).

- No singular contributions are therefore generated by (5.1.3), (5.1.7) and (5.1.8), when the discontinuity in p_0 is taken.

- The last and more complicated term involves the discontinuity in p_0 of (5.1.9), that is

$$-i\mathrm{disc}_{p_0}(5.1.9) = +i(1-s\frac{p_0}{p})(1-s'\frac{p_0'}{p'})\ \mathrm{disc}_{p_0}\sum_{i,j=0}^{3}\left(\sum_{k=-2}^{+1} a_{ij}^{k}\ (2Q{\cdot}P)^k\right)F_iF_j \quad (5.1.22)$$

Now this is simple also, because the a_{ij}^k of (5.1.17)-(5.1.21) are polynomials in p_0, and because $\delta(2Q{\cdot}P)$ has no support in the integration domain. Moreover, one has $\mathrm{disc}_{p_0}F_1 = \mathrm{disc}_{p_0}F_2 = 0$, whereas $\mathrm{disc}_{p_0}F_0$ is restricted to $= \pm i\pi\Theta(-P^2)/Z$ because, as stated above, $\delta(2Q{\cdot}P)$ has no support. The discontinuity of F_3 is restricted to $\pm i\pi Z\Theta(-P^2)/(p_0^2-2pxp_0+p^2)^2$ because, as demonstrated below, Eq.(5.2.38), $\delta(p_0^2-2pxp_0+p^2)$ and $\delta(p_0'-\omega_{s'}(p'))$ are incompatible constraints. One gets eventually

$$
\begin{aligned}
-i\mathrm{disc}_{p_0}(5.1.9) = \mp(1-s\frac{p_0}{p})(1-s'\frac{p_0'}{p'})\Big\{ (a_{02}^0)\ \frac{\pi\Theta(-P^2)}{Z}F_2 \\
+(\ a_{03}^{-1}Z^{-1}+a_{03}^0+a_{03}^1Z^1\ )\ (\ \frac{\pi\Theta(-P^2)}{Z}F_3+\frac{\pi\Theta(-P^2)}{(p_0^2-2pxp_0+p^2)^2}\ln\frac{P'^2}{P^2}\ ) \\
+(\ a_{23}^{-2}Z^{-2}+a_{23}^{-1}Z^{-1}+a_{23}^0+a_{23}^1Z^1\ )\ \frac{\pi\Theta(-P^2)ZF_2}{(p_0^2-2pxp_0+p^2)^2} \\
+2(\ a_{33}^{-2}Z^{-2}+a_{33}^{-1}Z^{-1}+a_{33}^0+a_{33}^1Z^1\ )\ \frac{\pi\Theta(-P^2)ZF_3}{(p_0^2-2pxp_0+p^2)^2}\Big\}
\end{aligned}
\tag{5.1.23}
$$

For the same reasons as before, it should be clear that when plugged back into (3.7), these terms, over (3.16), do not induce any singular behavior of the subsequent integrations on x, p_0 and p.

B. The case of $W_2^{(2)}(P,P')$

We now come to the last and most tedious angular integration, the one defining the function $W_2^{(2)}(P,P')$ of (5.2). Writing it as

$$
W_2^{(2)}(P,P') = \int_0^1 \mathrm{d}s\int_0^1 \mathrm{d}s'\int\frac{\mathrm{d}\widehat{K}}{4\pi}\int\frac{\mathrm{d}\widehat{K}'}{4\pi}\frac{1-2\widehat{K}^i\widehat{K}'_i+\widehat{K}^i\widehat{K}^j\widehat{K}'_i\widehat{K}'_j}{(\widehat{K}{\cdot}R(s)+i\epsilon)^2(\widehat{K}'{\cdot}R(s')+i\epsilon)^2}
\tag{5.2.1}
$$

it is possible to add and subtract a $+1$ in the numerator of (5.2.1), to get

$$
\begin{aligned}
W_2^{(2)}(P,P') = 2\int_{\widehat{K}}\int_{\widehat{K}'}\ \frac{\widehat{K}\cdot\widehat{K}'}{(\widehat{K}{\cdot}P+i\epsilon)(\widehat{K}{\cdot}P'+i\epsilon)(\widehat{K}'{\cdot}P+i\epsilon)(\widehat{K}'{\cdot}P'+i\epsilon)}-\frac{1}{Z^2}\ln^2\frac{P'^2}{P^2} \\
+\int_0^1 \mathrm{d}s\int_0^1 \mathrm{d}s'\int\frac{\mathrm{d}\widehat{K}}{4\pi}\int\frac{\mathrm{d}\widehat{K}'}{4\pi}\frac{\widehat{K}^i\widehat{K}^j\widehat{K}'_i\widehat{K}'_j}{(\widehat{K}{\cdot}R(s)+i\epsilon)^2(\widehat{K}'{\cdot}R(s')+i\epsilon)^2}
\end{aligned}
\tag{5.2.2}
$$

In the first line, the double angular integral is the one appearing already in (5.1.9), which as we have just seen, causes no singularity problem, and so is also the case of the second term. One can accordingly focus on the new, third term in the second line of (5.2.2).

This new term can be dealt with the help of the angular identity [12] ($R(s) = P + sQ$)

$$\int \frac{\mathrm{d}\widehat{K}}{4\pi} \; \frac{\widehat{K}^i\widehat{K}^j}{(\widehat{K}\cdot R(s)+i\epsilon)^2} = -\frac{g^{ij}}{r^2}Q_1(\frac{r_0}{r}) - \frac{r^i r^j}{r^2}\left(\frac{3}{r^2}Q_1(\frac{r_0}{r}) - \frac{1}{R^2(s)+i\epsilon r_0}\right) \tag{5.2.3}$$

an euclidean version of which can be found in [6]. When using that identity, one finds for the third term of (5.2.2), a sum of five fairly complicated contributions

$$\begin{aligned} W_2^{(2)}(P,P') \; \ni \; & -3\left(\int_0^1 \mathrm{d}s \; \frac{Q_1(R(s))}{r^2(s)}\right)^2 + 2\left(\int_0^1 \mathrm{d}s \; \frac{Q_1(R(s))}{r^2(s)}\right)\left(\int_0^1 \frac{\mathrm{d}s'}{R^2(s')+i\epsilon r_0(s')}\right) \\ & +9\int_0^1 \mathrm{d}s \; \frac{Q_1(R(s))}{r^2(s)} \int_0^1 \mathrm{d}s' \; [\widehat{r}(s)\widehat{r}(s')]^2 \frac{Q_1(R(s'))}{r^2(s')} - 6\int_0^1 \mathrm{d}s \; \frac{Q_1(R(s))}{r^2(s)} \int_0^1 \mathrm{d}s' \; \frac{[\widehat{r}(s)\cdot\widehat{r}(s')]^2}{R^2(s')+i\epsilon r_0(s')} \\ & + \int_0^1 \frac{\mathrm{d}s}{(R^2(s)+i\epsilon r_0(s))} \int_0^1 \mathrm{d}s' \; \frac{[\widehat{r}(s)\cdot\widehat{r}(s')]^2}{(R^2(s')+i\epsilon r_0(s'))} \end{aligned} \tag{5.2.4}$$

In order to express any of the five terms composing (5.2.4), and besides the definitions (5.1.12)-(5.1.16), the following integrals are needed :

$$\begin{aligned} I_3 = \int_0^1 \mathrm{d}s \; \frac{\ln X(s)}{r^3(s)} &= \frac{1}{qp^2(1-x^2)}\left((q+px)\frac{\ln X'}{p'} - px\frac{\ln X}{p}\right) \\ & - \frac{ZF_0}{qp^2(1-x^2)} + 2p_0F_3 + 2q\frac{F_2-P^2F_3}{Z} \end{aligned} \tag{5.2.5}$$

$$\begin{aligned} I_3' = \int_0^1 s\mathrm{d}s \; \frac{\ln X(s)}{r^3(s)} &= \frac{1}{pq^2(1-x^2)}\left(p\frac{\ln X}{p} - (p+qx)\frac{\ln X'}{p'}\right) \\ & + \frac{x}{pq^2(1-x^2)}ZF_0 - 2\frac{p^2}{q}F_3 + 2(p_0-2px)\frac{F_2-P^2F_3}{Z} \end{aligned} \tag{5.2.6}$$

where

$$X(s) = \frac{r_0(s)+r(s)}{r_0(s)-r(s)} \; , \quad X = X(0) = \frac{p_0+p}{p_0-p} \; , \quad X' = X(1) = \frac{p_0'+p'}{p_0'-p'} \tag{5.2.7}$$

Then, it is possible to give the final expression for the second term of (5.2.4). It is

$$2F_0\int_0^1 \mathrm{d}s \; \frac{Q_1(R(s))}{r^2(s)} = \frac{F_0}{pq(1-x^2)}\left((p_0x-p+\frac{Z}{2p})\frac{\ln X'}{p'} - (p_0x-p)\frac{\ln X}{p}\right) - \frac{1}{2}\frac{(ZF_0)^2}{p^2q^2(1-x^2)} \tag{5.2.8}$$

and also, for the first term of (5.2.4) :

$$\begin{aligned} & -3\left(\int_0^1 \mathrm{d}s \; \frac{Q_1(R(s))}{r^2(s)}\right)^2 \\ & = \frac{-3}{4p^2q^2(1-x^2)^2}\left(-\frac{1}{2}\frac{Z^2F_0}{pq} + (p_0x-p+\frac{Z}{2p})\frac{\ln X'}{p'} - (p_0x-p)\frac{\ln X}{p}\right)^2 \end{aligned} \tag{5.2.9}$$

- The fifth term of (5.2.4) can be cast into the form

$$\sum_{i,j=0}^{3}\left(\sum_{k=-2}^{+1} b_{ij}^{k}\ Z^{k}\right) F_i F_j \tag{5.2.10}$$

where the non vanishing b_{ij}^k are given by the array

$$b_{00}^{0}=1 \tag{5.2.11}$$

$$b_{22}^{-2}=2q^2p^2(1-x^2) \tag{5.2.12}$$

$$b_{33}^{-2}=2q^2p^2(1-x^2)(P^2)^2\ ,\quad b_{33}^{-1}=-4qp^3x(1-x^2)P^2\ ,\quad b_{33}^{0}=2p^4(1-x^2) \tag{5.2.13}$$

$$b_{03}^{0}=-2p^2(1-x^2) \tag{5.2.14}$$

$$b_{23}^{-2}=-4q^2p^2(1-x^2)P^2\ ,\quad b_{23}^{-1}=4qp^3x(1-x^2) \tag{5.2.15}$$

- The fourth term of (5.2.4) reads

$$\begin{aligned}
&-6F_0\left\{-\frac{1}{2}F_2+\frac{1}{2q}(\frac{q+px}{p'^2}-\frac{x}{p})+\frac{p_0}{2}I_3+\frac{q}{2}I_3'-\frac{p_0p^2(1-x^2)}{2}I_5-\frac{qp^2(1-x^2)}{2}I_5'\right\}\\
&\quad -6p^2(1-x^2)F_3\left\{-\frac{1}{p'^2}+(px-\frac{p_0}{2})I_3-\frac{q}{2}I_3'+\frac{p^2}{2q}ZI_5+pq\left(p(1-x^2)+\frac{x}{2q}Z\right)I_5'\right\}\\
&\quad -12qp^2(1-x^2)\frac{F_2-P^2F_3}{Z}\left\{\frac{1}{2qp}(\frac{p}{p'^2}-\frac{1}{p})+\frac{1}{2}I_3+\frac{p(p_0x-p)}{2}I_5+\frac{Z}{4}I_5'\right\}
\end{aligned} \tag{5.2.16}$$

where two extra more complicated integrals are needed :

$$I_5=\int_0^1 \mathrm{d}s\ \frac{\ln X(s)}{r^5(s)}\ ,\quad I_5'=\int_0^1 s\mathrm{d}s\ \frac{\ln X(s)}{r^5(s)} \tag{5.2.17}$$

One finds

$$\begin{aligned}
I_5=&\frac{1}{3qp^2(1-x^2)}\left(\frac{q+px}{p'^3}\ln X'-\frac{x}{p^2}\ln X\right)+\frac{2}{3qp^4(1-x^2)^2}\left(\frac{q+px}{p'}\ln X'-x\ln X\right)\\
&+\frac{4q}{3p^2(1-x^2)}\frac{F_2-P^2F_3}{Z}+\frac{2(p_0+px)}{3p^2(1-x^2)}F_3-\frac{2ZF_0}{3qp^4(1-x^2)^2}\\
&+\frac{2q}{3Z}\left(\int_0^1\frac{ds}{r^4(s)}+(p_0^2-2p_0px+p^2)\int_0^1\frac{ds}{R^2r^4(s)}\right)
\end{aligned} \tag{5.2.18}$$

$$\begin{aligned}
I_5'=&\frac{-1}{3q^2p(1-x^2)}\left(\frac{p+qx}{p'^3}\ln X'-\frac{1}{p^2}\ln X\right)-\frac{2x}{3q^2p^3(1-x^2)^2}\left(\frac{q+px}{p'}\ln X'-x\ln X\right)\\
&+\frac{2x}{3q^2p^3(1-x^2)^2}ZF_0-\frac{2x(p_0+px)}{3qp(1-x^2)}F_3-\frac{4x}{3p(1-x^2)}\frac{F_2-P^2F_3}{Z}\\
&+(\frac{1}{3q}-\frac{2qpx}{3Z})\int_0^1\frac{ds}{r^4(s)}\\
&-2p_0\frac{p_0^2-2p_0px+p^2}{3Z}\int_0^1\frac{ds}{R^2(s)r^4(s)}
\end{aligned} \tag{5.2.19}$$

with

$$\int_0^1 \frac{ds}{r^4(s)} = \frac{1}{2qp^2(1-x^2)}\left(\frac{px+q}{p'^2} - \frac{x}{p} + qF_2\right) \tag{5.2.20}$$

$$\int_0^1 \frac{sds}{r^4(s)} = \frac{1}{2pq^2(1-x^2)}\left(\frac{1}{p} - \frac{p+qx}{p'^2} - qxF_2\right) \tag{5.2.21}$$

and

$$\int_0^1 \frac{ds}{R^2(s)r^4(s)} = \frac{1}{q^2(p_0^2-2p_0px+p^2)^2}\Big\{Z^2F_3 + q^2\left(P^2 - \frac{2px}{q}Z\right)\int_0^1 \frac{ds}{r^4(s)} - q^2Z\int_0^1 \frac{sds}{r^4(s)}\Big\} \tag{5.2.22}$$

- Eventually, the third term of (5.2.4) is the more cumbersome one. It is

$$+9\int_0^1 \mathrm{d}s\ \frac{Q_1(R(s))}{r^2(s)}\int_0^1 \mathrm{d}s'\ [\widehat{r}(s)\cdot\widehat{r}(s')]^2\frac{Q_1(R(s'))}{r^2(s')} \tag{5.2.23}$$

An easier way to proceed consists in decomposing the intermediate integration, on s', into 3 pieces :

$$\int_0^1 \mathrm{d}s'\ [\widehat{r}(s)\cdot\widehat{r}(s')]^2\frac{Q_1(R(s'))}{r^2(s')} = (px+qs)^2\int_0^1 \frac{ds'}{r^2(s')}Q_1(s')$$
$$+p^2(1-x^2)(p^2-q^2s^2)\int_0^1 \frac{ds'}{r^4(s')}Q_1(s') + 2qp^2(1-x^2)(px+qs)\int_0^1 \frac{s'ds'}{r^4(s')}Q_1(s') \tag{5.2.24}$$

- The contribution to the third term of (5.2.4) coming from the 1st term of (5.2.24) is

$$\frac{9}{2}(p_0I_3+qI'_3-2F_2)\Big\{-\frac{F_2}{2}+\frac{1}{2q}(\frac{q+px}{p'^2}-\frac{x}{p})+\frac{p_0}{2}I_3+\frac{q}{2}I'_3 - \frac{p_0p^2}{2}(1-x^2)I_5 - \frac{qp^2}{2}(1-x^2)I'_5\Big\} \tag{5.2.25}$$

- The contribution to the third term of (5.2.4) coming from the second term of (5.2.24) is

$$\frac{9}{2}\left(p^2(1-x^2)(p_0I_5+qI'_5)+\frac{x}{pq}-\frac{q+px}{qp'^2}-F_2\right)\Big\{-\frac{1}{p'^2}+(px-\frac{p_0}{2})I_3-\frac{q}{2}I'_3 + \frac{p^2}{2q}ZI_5 + qp\left(p(1-x^2)+\frac{x}{2q}Z\right)I'_5\Big\} \tag{5.2.26}$$

- The contribution to the third term of (5.2.4) coming from the third term of (5.2.24) is

$$\frac{9}{2}\left(p^2(1-x^2)\left(I_3-p^2I_5+(\frac{Z}{2}-qpx)I'_5\right)-\frac{q+px}{p'^2}+pxF_2\right) \times\Big\{\frac{1}{q}(\frac{1}{p'^2}-\frac{1}{p^2})+I_3+p(xp_0-p)I_5+\frac{Z}{2}I'_5\Big\} \tag{5.2.27}$$

This shows how incredibly complicated is the exact calculation of an angular function like $W_2^{(2)}(P,P')$.

C. Collinear singularities

Let us recall Eq.(3.7) where the counterpart $W_2^{(2)}(P,P')$ of (5.2.2) is now substituted for the whole $W^{(2)}(P,P')$ of (3.6). The corresponding contribution to Im $\Pi_R^{(\star,\star;2)}(Q)$ one has to examine is

$$\pi e^2 m^4 \int \frac{\mathrm{d}^4 P}{(2\pi)^4}(1-2n_F(p_0)) \sum_{s,s'=\pm 1} \Big\{ -2\pi \beta_s^{(c)}(P)\beta_{s'}^{(p)}(P')W_2^{(2)}(P,P')$$
$$+ \Big(\alpha_s(P)\beta_{s'}^{(p)}(P') + \alpha_{s'}(P')\beta_s^{(p)}(P)\Big) \Big(-i \ \mathrm{disc}_{p_0} W_2^{(2)}(P,P')\Big)\Big\} \qquad (5.2.28)$$

Because it is simpler, we begin with analyzing the second term in the curly bracket of (5.2.28). In the original derivation of the hot QCD collinear singularity problem, it is this term which was responsible for a logarithmic singularity, [5].

Out of $W_2^{(2)}(P,P')$, or (5.2.4), and over the integration range (3.16), all contributions lead to integrals of form

$$C^{st} \sum_{s,s'} \int^q \frac{p^2 \mathrm{d}p}{(2\pi)^2} \int_{-p}^{p} \frac{\mathrm{d}p_0}{2\pi}(1-2n_F(p_0)) \ \alpha_s(p_0,p)$$
$$\times \int_{-1}^{p_0/p} \mathrm{d}x \, \delta\left(q+p_0-\omega_{s'}(p'(x))\right) \, (\omega_{s'}^2(p'(x)) - p'^2(x)) \, \frac{1}{(1-x^2)^a} \, \mathrm{disc}_{p_0} \, H(q,p,p_0;x) \qquad (5.2.29)$$

where (3.11) has been used, and where the power a is in the set $\{0,1,2\}$. Likewise, a function $H(q,p,p_0;x)$ stands for any of the functions that can be identified out of Eqs.(5.2.8), (5.2.9), (5.2.10)-(5.2.15,), (5.2.16), and (5.2.25)-(5.2.27). Then, inspection shows that the functions generically denoted by $H(q,p,p_0;x)$ can be decomposed into products of form

$$H(q,p,p_0;x) = \mathrm{Pol}\ (p_0;q,p,x) \times \frac{1}{(2Q\cdot P)^k}$$
$$\times \frac{1}{(p_0^2 - 2xp_0p + p^2)^b} \times (\ln \frac{p_0+p}{p_0-p})^c \times (\ln \frac{p'_0+p'}{p'_0-p'})^{c'} \qquad (5.2.30)$$

where Pol $(p_0;q,p,x)$, a polynomial in p_0, admits a Taylor series expansion in x, and where the integer powers k,b,c,c' are such that

$$0 \le k \le 2\ , \quad 0 \le b \le 4\ , \quad 0 \le c, c' \le 2 \qquad (5.2.31)$$

In view of the decomposition (5.2.30), the discontinuity in p_0 of any function $H(q,p,p_0;x)$ splits into a sum of four terms, any of them proportional to one only of the following list of discontinuities

$$\mathrm{disc}_{p_0} \ \mathrm{Pol}\ (p_0;q,p,x) = 0 \qquad (5.2.32)$$

$$\mathrm{disc}_{p_0}\ \frac{1}{2Q\cdot P} = -2i\pi\delta(2Q\cdot P) \tag{5.2.33}$$

$$\mathrm{disc}_{p_0}\ \frac{1}{(2Q\cdot P)^2} = 2i\pi\delta'(2Q\cdot P) \tag{5.2.34}$$

$$\mathrm{disc}_{p_0}\ \frac{1}{p_0^2-2xp_0p+p^2} = -2i\pi\delta(p_0^2-2xp_0p+p^2) \tag{5.2.35}$$

$$\mathrm{disc}_{p_0}\ \frac{1}{(p_0^2-2xp_0p+p^2)^b} = -2i\pi\frac{(-1)^{b-1}}{(b-1)!}\delta^{(b-1)}(p_0^2-2xp_0p+p^2) \tag{5.2.36}$$

$$\mathrm{disc}_{p_0}\ (\ln\frac{p_0+p}{p_0-p})^c = -ci\pi\Theta(-P^2)(\ln\frac{p_0+p}{p_0-p})^{(c-1)} \tag{5.2.37}$$

(i) -The first case, (5.2.32), is trivial.

(ii) -Terms proportional to the second and third discontinuities, (5.2.33) and (5.2.34), give zero because of the incompatibility of $\delta(2Q\cdot P)$ and $\delta\left(p_0'-\omega_{s'}(p'(x))\right)$.

(iii) -Terms proportional to the fourth and fifth discontinuities, (5.2.35) and (5.2.36). At real p_0-energies, the $\delta(p_0^2-2xp_0p+p^2)$ -constraint is satisfied at $x=+1$ where $p_0=p$, and at $x=-1$ where $p_0=-p$, the latter case excluded by (3.16). Now, a cogent argument, approximation-free and valid at $x=\pm1$, is the following: At $x=\pm1$, one has $p'(x=\pm1)=q\pm p$, and so

$$\delta(q+p_0-\omega_{s'}(p')) = \delta(q\pm p-\omega_{s'}(q\pm p)) \tag{5.2.38}$$

that has no support in the integration range (and beyond) because, for all $s'=\pm1$, the relation $\omega_{s'}(q\pm p)-(q\pm p)>0$ holds, in virtue of Fig.3. The two constraints are incompatible, and the corresponding contributions are zero.

(iv) -Terms proportional to the last discontinuity, (5.2.37) involve both a $\Theta(-P^2)$ and a $\delta(p_0'-\omega_{s'}(p'(x)))$ distribution: They turn out to be identical to the terms related to the first piece of the curly bracket of (5.2.28) that can be analyzed now.

The first piece in the curly bracket of (5.2.28) requires more care. One can start from an expression similar to (5.2.29),

$$C^{st}\sum_{s,s'}\int^q\frac{p^2\mathrm{d}p}{(2\pi)^2}\int_{-p}^{p}\frac{\mathrm{d}p_0}{2\pi}(1-2n_F(p_0))\ \beta_s^{(c)}(p_0,p)$$
$$\times\int_{-1}^{p_0/p}\mathrm{d}x\ \delta\left(q+p_0-\omega_{s'}(p'(x))\right)\ (\omega_{s'}^2(p'(x))-p'^2(x))\frac{H(q,p,p_0;x)}{(1-x^2)^a} \tag{5.2.39}$$

with the same set of functions $H(q,p,p_0;x)$ as defined in (5.2.30). In this way, *potential* collinear singularities are emphasized, as terms proportional to $1/(1-x^2)^a$, with $a\in\{1,2\}$.

For example, such is the case of integrals I_3, I'_3, I_5 and I'_5, all of them able to generate collinear singularities at $x = \pm 1$, $p_0 = \pm p$.

Clearly, a closer inspection of $H(q, p, p_0; x)$-functions is in order, and more to the point, a regrouping of terms proportional to the potentially dangerous factors of $1/(1 - x^2)^a$.

- Then, one finds that the integrals I_3 and I'_3 are not on the order of $1/(1 - x^2)$, but are regular functions of x at $x = \pm 1$,

$$\{I_3, I'_3\}_{|x=\pm 1} = \mathcal{O}\left((1 \mp x)^0\right) + \dots \tag{5.2.40}$$

where the dots stand for higher orders in $(1 \mp x)$.

- Likewise, and thanks to the very same compensations as those at work in the case of I_3 and I'_3, one finds that I_5 and I'_5 are not on the order of $(1 - x^2)^{-2}$, but at worse, on the order of $(1 - x^2)^{-1}$,

$$\{I_5, I'_5\}_{|x=\pm 1} = \mathcal{O}\left((1 \mp x)^{-1}\right) + \mathcal{O}\left((1 \mp x)^0\right) + \dots \tag{5.2.41}$$

The functions I_5 and I'_5 also depend on the integrals (5.2.20) and (5.2.21) which, like I_3 and I'_3, are regular functions of x, at $x = \pm 1$. In the case of (5.2.20) for example, one gets,

$$\begin{aligned}\frac{1}{2qp^2(1-x^2)}\left(\frac{px+q}{p'^2} - \frac{x}{p} + qF_2\right)_{|x=\pm 1} &= \frac{1}{4qp^2}\frac{1}{1 \mp x}\Bigg\{\frac{\pm p + q}{(q \pm p)^2} - \frac{\pm 1}{p} \\ &+ \frac{q}{p(p \pm q)} + \mathcal{O}(1 \mp x) + \dots = 0 + \mathcal{O}(1 \mp x) + \dots\Bigg\}\end{aligned} \tag{5.2.42}$$

The same applies to (5.2.21), and another similar example of potential collinear singularity compensation will be given below.

- Eventually, a third useful property is that the four combinations

$$pI_n + qxI'_n\ , \quad pxI_n + qI'_n\ , \quad n \in \{3, 5\} \tag{5.2.43}$$

are able to decrease by one unit the power a of any $1/(1 - x^2)^a$- contributions appearing in I_n and I'_n (at $n = 3$, combinations (5.2.43) are on the order of $(1 - x^2)$, in view of (5.2.40)).

One is now in a position so as to analyze the contributions to (5.2.39) of any of the five terms composing $W_2^{(2)}(P, P')$. For the previous form of (5.2.39), it may be more convenient now, to substitute the expression

$$\begin{aligned}&C^{st}\sum_{i=1}^{5}\sum_{s,s'}\int^q \frac{p^2\mathrm{d}p}{(2\pi)^2}\int_{-p}^{p}\frac{\mathrm{d}p_0}{2\pi}(1 - 2n_F(p_0))\ \beta_s^{(c)}(p_0, p) \\ \times &\int_{-1}^{p_0/p}\mathrm{d}x\ \delta\left(q + p_0 - \omega_{s'}(p'(x))\right)\ \left(\omega_{s'}^2(p'(x)) - p'^2(x)\right)\ T^{(i)}(q, p, p_0; x)\end{aligned} \tag{5.2.44}$$

where the $T^{(i)}(q,p,p_0;x)$ denote the five contributions displayed in (5.2.8), (5.2.9), (5.2.10), (5.2.16) and (5.2.25)-(5.2.27).

- $T^{(5)}$, the fifth term of (5.2.4), given in (5.2.10) and the array of coefficients (5.2.11)-(5.2.15), is a linear combination of regular functions of x, p_0 and p over the full integration range (3.16), and its contribution to (5.2.44) is singularity free.

- $T^{(2)}$, the second term of (5.2.4), given in (5.2.8), appears singular in the collinear regime of $x=\pm 1$. But it is not so, and at $x=-1$ (as well as at $x=1$) the right hand side of (5.2.8) behaves like

$$\frac{F_0(x=-1)}{2pq(1+x)}\left(\ln\frac{p_0'+p'}{p_0'-p'}+\ln\frac{p_0+p}{p_0-p}-\ln\frac{P'^2}{P^2}\right)=\mathcal{O}(\frac{1}{1+x})(\ln\frac{p_0+p}{p_0'-p'})_{|x=-1}+\mathcal{O}\left((1+x)^0\right)$$
$$=\mathcal{O}(\frac{1}{1+x})\left(\ln\frac{p_0+p}{q+p_0-(q-p)}\right)+\mathcal{O}\left((1+x)^0\right)=\mathcal{O}\left((1+x)^0\right) \qquad (5.2.45)$$

and therefore, its integration over x is collinear singularity free.

- So is also the contribution to (5.2.44) of $T^{(1)}$, the first term of (5.2.4). As displayed by (5.2.9), in effect, this term is the square power of the previous one. In view of (5.2.45), its integration on x is collinear singularity free either.

- $T^{(4)}$, the fourth term of (5.2.4) is given by (5.2.16), and Eqs.(5.2.40), (5.2.41) guarantee that this term lead to collinear singularity free contributions to (5.2.44).

- $T^{(3)}$, the third term of (5.2.4) is given by (5.2.25), (5.2.26) and (5.2.27). The first part, (5.2.25), leads to a regular contribution in virtue of (5.2.40) and (5.2.41). The second part, (5.2.26), leads to a regular contribution in virtue of (5.2.40) and (5.2.41), and also in virtue of the first combination of (5.2.43) taken at $n=5$. The third part, (5.2.27), leads to a regular contribution in virtue of (5.2.40) and (5.2.41), and also in virtue of the two combinations of (5.2.43) taken at $n=5$; to wit, from the second line:

$$\cdots+p(xp_0-p)I_5+\frac{Z}{2}I_5'=\cdots+p_0(pxI_5+qI_5')-p(pI_5+qxI_5')=\cdots+\mathcal{O}\left((1-x^2)^0\right) \quad (5.2.46)$$

where (5.2.41) has been used.

To summarize, relevant regroupings of the initial $H(q,p,p_0;x)$-functions are able to display a full compensation of all *potential* collinear singularities (both at $x=-1$ and at $x=1$): Certainly, these fine tuning compensations, taking place among so many terms, do not show up by pure chance, and clearly, they support the reliability of the calculations that are presented here.

As illustrated in Appendix B, collinear singularities would pass from *potential* to *actual* upon integration on x, p_0 and p, and not upon integration on x alone.

At this point, an important remark is in order.
In the range (3.16), an inspection of the remaining integrations has not revealed any further difficulties: The angular functions $W^{(1)}$ and $W^{(2)}$ display singular behaviours at the light cone $P^2 = 0$ (of the logarithmic type for example), that do not compromise the regular character of the full integration over (3.16). Now, in this respect, it matters to emphasize that a complete compensation of potential or actual collinear singularities is of utmost importance. As displayed through Appendix B, in effect, terms of $1/(1-x^2)^a$ do not yield any collinear singularity *as such*. Instead, out of the remaining p_0 and p-integrations, potential collinear singularities generate further logarithmic and power-law singularities, as well as products of them.

What is more, these further fake singularities can be proven to receive no screening/removal at all from an improved HTL-effective action resumming *asymptotic thermal masses* along both bosonic and fermionic lines, [8]. This very unusual circumstance, fully understandable though, is demonstrated in Appendix B.

These examples therefore, are highly suggestive of the crucial importance of potential/actual collinear singularity compensations, if any. Missing the completeness of collinear singularity compensations, results into severe further troubles: As suggested in Appendix B, the collinear-enhancement mechanism and the related loop-expansion breaking, are very likely nothing else than some of these troubles.

VI. CONCLUSION

The hot QCD collinear singularity problem had to be revisited entirely, and were it not for the tedious calculations this revisitation requires, the task could have been achieved sooner.

As we have seen in effect, the thorough evaluation of entwined angular averages is very complicated and a lot of patient checkings is needed. This is the more so as, to our knowledge (and ability) at least, no mathematical program is really able to yield the full results of Section 5.2. Getting them however, is the price to be paid in order to fix definitely that 16

years old issue (experience shows in effect, that attempts at guessing the essential features of such complicated objects as those entwined angular averages, are doomed to failure).

Our results can be summarized as follows.

- In the first place, having proceeded, within the correct sequence, to a most careful analysis of the 1- and 2- effective vertex diagrams relevant to the soft photon emission rate out of a Quark-Gluon Plasma at thermal equilibrium, we claim that the corresponding emission rate is singularity-free. The hot QCD collinear singularity problem simply doesn't exist, and in textbooks, should no longer be presented as a serious obstruction to the Resummation Program.

As pinned up in [11], the 1994's- famous divergent result, [5], is due to erroneous manipulations due to the fuzzy distinction made in our formalisms, between the prescriptions of *discontinuity* and *imaginary part*. Whereas the latter commutes with an integration process, by integration's linearity, the former does not, in general, because it is defined by a limiting procedure, [11]. Now, the prevalence of the discontinuity prescription over the imaginary part one has been advocated in Ref.[17].

- Appendix B has revealed instructive aspects. To summarize, if for some reason, a complete compensation of (potential or actual) collinear singularities is missed, then, the same drawbacks occur, as encountered by the improved effective perturbation theory:

- Resummed asymptotic thermal masses, bosonic and fermionic, do not provide enough screening, and, due to power-law collinear-induced singularities, a full leading order emission rate calculation requires higher order diagrams.

- Moreover, the required extra diagrams may clearly depend on the regulators that are choosen in order to quantify the collinear-induced singularities of the original diagrams. An unavoidable arbitrariness is thus introduced in the emission rate leading order completion, supposing under control that extra diagrams are determined at the exclusion of any others.

- And last but not least, extra diagrams are also expected to compensate for original singularities, [18]. But we have seen here, how very peculiar to the diagram under consideration, are the collinearly generated singularities. Now, infrared/collinear cancellations between diagrams of different topologies, [18], in a non-abelian context, what is more, [19], is a highly non-trivial conjecture, if not an exceptional one: If that possibility can be thought of as reliable, at least so long as the stronger infrared singularities are concerned, [20], there

is no guarantee whatsoever that it could be so in the case of sub-leading ones; quite on the contrary, [21].

- It is therefore suggested that all of the above long known difficulties, express an incomplete compensation of initial collinear singularities, and nothing else. This is the more likely so, as the calculations presented here are able, among so many terms, to exhibit a fine tuning compensation of all of the possible collinear singularities: A cogent enough result, which can not happen just by chance.

Accordingly, right from hot QCD first principles, the finite contributions that remain provide us with a sound, reliable basis for a complete leading order estimate of the soft photon emission rate out of a QGP at thermal equilibrium, [22]. This perspective should be of interest in view of RHIC and LHC experimental runs.

Appendix A

The second line of Eq.(4.5) defines $F(p_0)$, the function

$$F(p_0) = \sum_{s'=\pm 1} \int_{-1}^{\frac{p_0}{p}} \mathrm{d}x \, \frac{\delta\left(\omega_{s'}(p'(x) - q - p_0\right)}{2q(\omega_{s'}(p'(x)) - (q+px))} \, (\omega^2_{s'}(p'(x)) - p'^2(x))(1 - s'\frac{\omega_{s'}(p'(x))}{p'(x)}) \quad (A.1)$$

where the denominator has been expressed as $2Q{\cdot}P'$. We recall that $p'^2(x) = p^2 + 2pqx + q^2$ and $\vec{q}{\cdot}\vec{p'} = q(q+px)$. In order to see the regular character of $F(p_0)$ in a neighborhood of $p_0 = p$, one may expand the constraint of $\delta\left(\omega_{s'}(p'(x) - q - p_0\right)$,

$$\delta\left(\omega_{s'}(p'(x) - q - p_0\right) = \delta\left(\omega_{s'}(p'(x) - (q+p)\right) - (p - p_0)\frac{\mathrm{d}}{\mathrm{d}(q+p)}\delta(q + p - \omega_{s'}(p'(x))) + .. \quad (A.2)$$

where the dots stand for higher order corrections in $(p - p_0)$, and obtain

$$\begin{aligned} F(p_0) = \sum_{s'=\pm 1} & \left\{ \int_{-1}^{+1} \mathrm{d}x \; \delta\left(q + p - \omega_{s'}(p'(x))\right) \, (1 - s'\frac{q+p}{p'(x)}) \right. \\ -(p - p_0)\frac{\mathrm{d}}{\mathrm{d}(q+p)} & \left. \int_{-1}^{+1} \mathrm{d}x \; \delta(q + p - \omega_{s'}(p'(x))) \, (1 - s'\frac{q+p}{p'(x)}) + .. \right\} \end{aligned} \quad (A.3)$$

Note that instead of $P^2 < 0$, which in addition to $p_0 > -p$, (3.16), would also preclude any risk of potentially singular behavior at $p_0 = p$, one allows for $P^2 \leq 0$ in view of the step-function $\Theta(-P^2)$ appearing in (3.10). This is equivalent to $P'^2 - 2Q{\cdot}P' \leq 0$, which at $p'_0 = q + p_0 = \omega_{s'}(p'(x))$ is guaranteed, provided the inequality $q - p \leq \omega_{s'}(p'(x)) \leq q + p$

be satisfied. By picking up a value of x smaller than 1 (at $x = 1$ in effect, any term in the expansion (A.3) would be zero because of the argument of Eq.(5.2.38), the latter inequality allows for the constraint of $\delta(q+p-\omega_{s'}(p'(x)))$ to have a non-empty support in the integration domain (3.16).

Appendix B

In this appendix, it is assumed that a full compensation of collinear singularities is not obtained, so that in (5.2.39), some function $H(q,p,p_0;x)$ remains, hereafter denoted by $\mathcal{H}(q,p,p_0;x)$, whose behaviour at $x=-1$ does not compensate for the potentially dangerous factors of $1/(1-x^2)^a$. What happens then?

In order to examine the behavior of (5.2.39) in the collinear regime of $x \simeq -1$, one can rely on the expansion [6]

$$\omega_{s'}(p'(x)) \simeq m\left(1 + s'\frac{p'(x)}{3m} + \mathcal{O}(\frac{p'}{m})^2\right) \;, \quad x \simeq -1 \qquad (B.1)$$

This expansion makes sense provided that $p'(x)/m << 1$, a condition that is met at $x \simeq -1$, in view of (3.16) and in view also, of the relatively narrow phase-space extension of the Resummation Program, [14, 15]. Without prejudice to our concern, $(B.1)$ allows us to replace in (5.2.39), the factor $\omega^2_{s'}(p'(x)) - p'^2(x)$ by the constant m^2, because one has, [6],

$$\frac{\omega^2_{s'}(p') - p'^2}{2m^2} \simeq \frac{1}{2} + s'\frac{p'}{m} \qquad (B.2)$$

The contribution to (5.2.39) of the collinear regime $x \simeq -1$ therefore reads as

$$\sim C^{st}\frac{9m^2}{q}\sum_{s,s'}\int^q \frac{p\mathrm{d}p}{(2\pi)^2}\left(\frac{3qp}{2(q-p)}\right)^a$$
$$\times \int_{p_{01}}^{+p}\frac{\mathrm{d}p_0}{2\pi}(1-2n_F(p_0))\,\beta_s^{(c)}(p_0,p)\,\mathcal{H}(q,p,p_0;x_0)(q+p_0-m)\left(\frac{1}{p_0-p_{01}} - \frac{1}{p_0-p_{02}}\right)^a \qquad (B.3)$$

where the two zeros p_{0i} are

$$p_{01} \simeq m - p - \frac{2}{3}(q-p) \;, \quad p_{02} \simeq m - p - \frac{4}{3}(q-p) \qquad (B.4)$$

and where, bearing on the angle selected by the constraint $\delta\left(\omega_{s'}(p'(x) - q - p_0\right)$, the condition

$$-1 \leq x_0 = -1 + \frac{9(p_0-p_{01})(p_0-p_{02})}{2qp} \qquad (B.5)$$

restricts the original p_0-range to the interval $[p_{01}, +p]$.

The ensuing integrations do not exist in the rigorous mathematical sense because of a pole at $p_0 = p_{01}$, and another one at $p = q$, both induced by a potential collinear singularity at $x = -1$. As is often the case at non-zero temperature, [15, 16], extra regularizations must be supplied. Let it be done by shifting the pole at p_{01} a small amount of λ, and the pole at $p = q$, a small amount of δq. Then, in $(B.3)$, two values of a come into play:

(i) At $a = 1$, and in the limit of vanishing regulators, $\lambda = 0$ and $\delta q = 0$, two logarithmically divergent contributions come out, on the order of

$$\mathcal{O}\left(\ln \frac{\delta q}{q}\right) + \mathcal{O}\left(\ln \frac{q}{\lambda}\right) \qquad (B.6)$$

(ii) At $a = 2$, to the two previous singular behaviours, $(B6)$, one must add singular contributions on the strength of

$$\mathcal{O}\left(\ln \frac{\delta q}{q} \times \ln \frac{q}{\lambda}\right) + \mathcal{O}\left(\frac{1}{\lambda}\right) + \mathcal{O}\left(\frac{1}{\delta q}\right) + \mathcal{O}\left(\frac{1}{\delta q} \times \ln \frac{q}{\lambda}\right) + \mathcal{O}\left(\frac{1}{\lambda} \times \ln \frac{q}{\delta q}\right) \qquad (B.7)$$

It is worth remarking that these singular behaviours are generated by the $x = -1$-collinear regime: Whereas a genuine collinear singularity, as such, does not appear, a *potentially* singular collinear behaviour is at the origin of the *actual* singular terms developed by the remaining integrations on p_0 and p.

(iii) An amazing feature revealed by this calculation is worth emphasizing. If one proceeds to improve the bosonic and fermionic HTL-effective actions in the sense of Ref.[8], providing gluon and quark fields with asymptotic thermal masses m_∞ and M_∞ respectively, then, no change is brought to the above situation. Contrarily to usual expectations, the singularities of $(B.6)$ and $(B.7)$ receive no screening from a resummation of asymptotic thermal masses.

This can be seen as follows. Asymptotic thermal masses will affect the effective quark propagators, the ${}^\star S_R(P)$ of (2.2), by substituing to (2.3) an improved version of the thermal self energy,

$$(2.3) \longrightarrow m^2 \, \frac{2}{\pi^2} \int_0^\infty \mathrm{d}\alpha \, \frac{\alpha \, e^\alpha}{e^{2\alpha} - 1} \int \frac{\mathrm{d}\widehat{K}}{4\pi} \left(\frac{\widehat{K}\!\!\!/}{\widehat{K}\cdot P + \frac{\mathrm{d}m}{\alpha}} + \frac{\widehat{K}\!\!\!/}{\widehat{K}\cdot P - \frac{\mathrm{d}m}{\alpha}} \right) \qquad (B.8)$$

where,

$$\mathrm{d}m = \frac{m_\infty^2 - M_\infty^2}{2T}, \quad m_\infty^2 = \frac{g^2 T^2 N}{6}, \quad M_\infty^2 = \frac{g^2 T^2 C_F}{8} \qquad (B.9)$$

And likewise, the effective photon-quark-quark vertex, the $\Gamma_\mu^{HTL}(P_\alpha, Q_\beta, P'_\delta)$ of (2.5) is improved in a similar way, [8],

$$C^{st}\int_0^\infty \mathrm{d}\alpha\, \frac{\alpha\, e^\alpha}{e^{2\alpha}-1}\int \frac{\mathrm{d}\widehat{K}}{4\pi}\left(\frac{\widehat{\not K}\ \widehat{K}^\mu}{(\ \widehat{K}\cdot P+\frac{\mathrm{d}m}{\alpha})(\ \widehat{K}\cdot P'+\frac{\mathrm{d}m}{\alpha})}+\frac{\widehat{\not K}\ \widehat{K}^\mu}{(\ \widehat{K}\cdot P-\frac{\mathrm{d}m}{\alpha})(\ \widehat{K}\cdot P'-\frac{\mathrm{d}m}{\alpha})}\right) \tag{$B.10$}$$

Surprisingly enough, in the latter case, factors of $(1-x^2)^{-a}$, at $a=1$ and $a=2$, are left the same as at $\mathrm{d}m=0$, and collinear singularities at $x=-1$ receive no screening. As inspection shows in effect (Section V.B), this is so because those singular factors come exclusively from the $r^{-3}(s)$, $r^{-4}(s)$ and $r^{-5}(s)$ pieces of the various functions to be integrated over $s\in[0,1]$: One has $r^2(s)=p^2+2pqxs+q^2s^2$, (5.1.12), and Tables show that the ensuing integrations come out proportional to inverse powers of $\Delta=4p^2q^2-(2pqx)^2$, [13]. Now, the function $r^2(s)$ itself comes from the scalar product $\vec{r}(s)\cdot\vec{r}(s)$, where the vector $\vec{r}(s)=\vec{p}+s\vec{q}$ is clearly unaffected by a shift of p_0 to $p_0\pm\mathrm{d}m/\alpha$, followed by an average over α.

Note that a full calculation only, was able to reveal such a fate, so as the related peculiar nature of the ensuing collinear singularities.

In the former case, quasi-particle poles, $\omega_s(p)$ of Eq.(3.8), are solutions of

$$p_0-sp-\frac{m^2}{2p}\left((1-s\frac{p_0}{p})\ln\frac{p_0+p}{p_0-p}+2s\right)=0 \tag{$B.11$}$$

and with $(B.8)$, will now become solutions of

$$p_0-sp-s\frac{m^2}{p}-\frac{1}{2}\left\{\left\langle\sum_{\eta=\pm1}\frac{m^2}{2p}(1-s\frac{p_0+\eta\frac{dm}{\alpha}}{p})\ln\frac{p_0+p+\eta\frac{dm}{\alpha}}{p_0-p+\eta\frac{dm}{\alpha}}\right\rangle\right\}=0 \tag{$B.12$}$$

where the notation $\langle\ldots\rangle$ has been introduced as a shorthand to mean

$$\left\langle F(\alpha)\right\rangle=\frac{2}{\pi^2}\int_0^\infty \mathrm{d}\alpha\ \frac{\alpha\ e^\alpha}{e^{2\alpha}-1}F(\alpha) \tag{$B.13$}$$

For this new equation to admit a new solution in the integration range of (3.16), $\widehat{\omega}_s(p)$, the new terms composing it cannot be an order of magnitude bigger than the remaining ones, and in view of $(B.9)$, this observation imposes $g/\alpha<1$, that is, α cannot be smaller than g. Then, considering the P'-fermionic line, relevant to the crossed possibilities of (3.10), after some algebra, it is possible to re-write $(B.12)$ as,

$$p'_0-s'p'(x)-s'\frac{m^2}{2p'}(1-s'\frac{p'_0}{p'})\ln\frac{p'_0+p'}{p'_0-p'}-s'\frac{m^2}{p'}\left(1+\frac{1}{2}s'(1-s'\frac{p'_0}{p'})\langle\frac{1}{\alpha}\rangle\frac{p'_0}{m}\cdot\frac{dm}{m}\right)+\cdots=0 \tag{$B.14$}$$

where the dots stand for higher orders in a small parameter development, the parameter $p\mathrm{d}m/\alpha P^2$. At $p_0 \pm p = \mathcal{O}(gT)$, this small parameter is on the order of $\mathrm{d}m/\alpha(p_0 \pm p) \simeq \mathcal{O}(g/6\alpha)$, with $g/\alpha < 1$. Note that in $(B.14)$, because of $\alpha > g$, we have now a slight modification of the average introduced in $(B.13)$,

$$\left\langle \frac{1}{\alpha} \right\rangle = \mathrm{C}^{\mathrm{st}}(g) \int_g^\infty \mathrm{d}\alpha\ \frac{\alpha\ e^\alpha}{e^{2\alpha}-1}\ \frac{1}{\alpha} = \frac{1}{2}\mathrm{C}^{\mathrm{st}}(g) \ln\frac{1}{g} = (\frac{2}{\pi^2}\ln\frac{1}{g})(1+\mathcal{O}(g)) \qquad (B.15)$$

A comparison of $(B.14)$ to $(B.11)$ shows that the former, whith respect to the same equation taken at $\mathrm{d}m = 0$, is modified an amount of relative magnitude

$$\frac{s'}{2}(1 - s'\frac{p'_0}{p'})\langle\frac{1}{\alpha}\rangle\frac{p'_0}{m}\cdot\frac{dm}{m} = -\frac{1}{\pi^2}\sqrt{\frac{2}{3}\frac{p'_0}{m}}\ g\ln g \equiv \varepsilon(\mathrm{d}m) = \mathcal{O}(g\ln g) \qquad (B.16)$$

which can still be taken as a small enough quantity.

Denoting by $\omega'_{s'}(p')$ the solutions to $(B.14)$ taken at $\mathrm{d}m = 0$, this suggests to look for solutions to $(B.14)$ under the form of

$$\widehat{\omega'}_{s'}(p') = \omega'_{s'}(p') + \delta\omega'_{s'}(p') \qquad (B.17)$$

and to analyze the consequence on the pole-location, coming from the new inherited constraint of $\delta(q + p_0 - \widehat{\omega'}_{s'}(p'(x)))$. Then, provided that the condition

$$\frac{\delta\omega'_{s'}}{\omega'_{s'} - p'} << 1 \qquad (B.18)$$

is satisfied, one finds

$$\delta\omega'_{s'}(p') = -2\varepsilon(\mathrm{d}m)\frac{s'm^2(\omega' - s'p')}{-p'(\omega' - s'p') - s'm^2 + 2(\omega' - s'p')^2\frac{m^2}{\omega'^2 - p'^2}} \qquad (B.19)$$

where ω' is a shorthand for $\omega'_{s'}(p'(x))$. Since $\omega'_{s'}(p'(x))$ complies with the expansion $(B.1)$ for x in a neighbourhood of -1, this expansion can be used in $(B.19)$ so as to get

$$\delta\omega'_{s'}(p') = -\varepsilon(\mathrm{d}m)\left(\frac{2s'm}{2-s'} + \mathcal{O}(\frac{p'}{m})\right) \qquad (B.20)$$

and we note that $(B.20)$ complies with $(B.18)$,

$$\frac{\delta\omega'_{s'}}{\omega'_{s'} - p'} \simeq \mathcal{O}(\varepsilon(\mathrm{d}m)) \qquad (B.21)$$

Since $\delta\omega'_{s'}(p')$ is proportional to m, the new constraint, $\delta(q + p_0 - \widehat{\omega'}_{s'}(p'(x)))$, will amount to re-define the pole at p_{01},

$$p_{01} \longrightarrow \widehat{p_{01}} \simeq m\left(1 - \frac{2s'}{2-s'}\varepsilon(\mathrm{d}m)\right) - p - \frac{2}{3}(q - p) \qquad (B.22)$$

with, obviously, the same conclusions $(B.6)$ and $(B.7)$ as at $\mathrm{d}m = 0$.

What can be learned out of this example? Apparently, that if a complete compensation of potential/actual collinear singularities is missed, then further singularities develop, and that a resummation of thermal asymptotic masses does not bring enough screening, to say the less. Also, since some of the generated singularities are power-law, what shouldn't come as a surprise (see Ref.[8]), it becomes obvious that, depending on the scale of the adopted regulators (the λ and δq), higher number of loop diagrams will be found to be on the same orders of magnitude as elementary ones.

Needless to emphasize that these difficulties are of course very similar to those which are known to plague the HTL- improved effective perturbation theory, [9]. Note also that, depending on the scale of the adopted regulators, the extra diagrams that will become necessary to complete the full soft photon emission rate leading order, will differ .. and in any case, will be hoped to cancel out the original singularities, [18], ..

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